TOPOLOGY - III, SOLUTION SHEET 8

Exercise 1.
(1) This follows from applying excision to the open sets U and $X - \{x\}$, which clearly cover X.
(2) By (1) we have that $H_k(M, M - \{x\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n - \{0\})$. By exercise 4 of sheet 7 this group is isomorphic to $\mathbb{Z}$ if $k = n$ and 0 otherwise.
(3) By (1) we can assume that G is a star graph with d edges, where d is the degree of the vertex v. Then G is contractible and $G - \{v\}$ is the disjoint union of d half-open line segments. Therefore the long exact sequence of relative homology yields that $H_k(G, G - \{v\}) \cong \mathbb{Z}^{d-1}$ for $k = 1$ and 0 otherwise.
(4) By (1) the relative homology of the curve C at the nodal point p is the same as the relative homology of a star graph with 4 edges. Therefore $H_1(C, C - \{p\}) \cong \mathbb{Z}^3$. Hence, it follows from (2) that C is not a topological manifold.

Exercise 2. Please refer to the part of the proof of Theorem 2.27 on page 130 in Hatcher's book for the solution of all three parts.